The Spectral Theorem

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Chapter 1

Introduction

The principal assertion of the spectral theorem is that every bounded normal operator T on a Hilbert space induces (in a canonical way) a resolution E of the identity on the Borel subsets of its spectrum $\sigma(T)$ and that T can be reconstructed from E by an integral. A large part of the theory of normal operators depends on this fact.

Chapter 2

The Reisz Theorem

- Destination: MeasureTheory/Integral/RieszMarkovKakutam/ComplexRMK
- Principal reference: Theorem 6.19 of [Walter Rudin, Real and Complex Analysis.][Rud87].

The main statement is:

If X is a locally compact Hausdorff space, then every bounded linear functional Φ on $C_0(X)$ is represented by a unique regular complex Borel measure μ, in the sense that

$$\Phi f = \int_X f \, d\mu$$

for every $f \in C_0(X)$. Moreover, the norm of Φ is the total variation of μ:

$$\|\Phi\| = |\mu|(X).$$

Definition 1 (Variation of a Vector-Valued Measure)**.** Let $(X, \mathcal{A})$ be a measurable space and let Y be a Banach space. For a vector-valued measure $\mu : \mathcal{A} \to Y$, the **variation** of μ is the set function $|\mu| : \mathcal{A} \to [0, +\infty]$ defined by

$$|\mu|(E) = \sup \left\{ \sum_{i=1}^{n} \|\mu(E_i)\|_Y : \{E_1, E_2, \dots, E_n\} \text{ is a finite partition of } E \text{ in } \mathcal{A} \right\}$$

for each $E \in \mathcal{A}$.

Equivalently, the above definition can be written as:

$$|\mu|(E) = \sup \left\{ \sum_{i=1}^{n} \|\mu(E_i)\|_Y : E_i \in \mathcal{A},\, E_i \cap E_j = \emptyset \text{ for } i \neq j,\, \bigcup_{i=1}^{n} E_i \subseteq E \right\}$$

Theorem 2 (Rudin 6.12 (polar representation of a complex measure))**.** *Let μ be a complex measure on a σ-algebra $\mathfrak{M}$ in X. Then there is a measurable function h such that $|h(x)| = 1$ for all $x \in X$ and such that*

$$d\mu = h \, d|\mu|. \tag{2.1}$$

Proof. This rather depends on how the integral with respect to a complex measure is defined. See [Rudin, Theorem 6.12] for details. □

Definition 3. Let X be a locally compact Hausdorff space. Associated to every bounded linear functional Φ on $C_0(X)$ we define a regular complex Borel measure μ which we call the Riesz Measure associated to Φ.

TO DO: insert details from the proof of the exact definition.

In order to prove the main result we divide the result into several smaller results.

Theorem 4 (Rudin 3.14)**.** *For $1 \leq p < \infty$, $C_c(X)$ is dense in $L^p(\mu)$.*

Proof. Define S as in Theorem 3.13. If $s \in S$ and $\varepsilon > 0$, there exists a $g \in C_c(X)$ such that $g(x) = s(x)$ except on a set of measure $< \varepsilon$, and $|g| \leq \|s\|_\infty$ (Lusin's theorem). Hence

$$\|g - s\|_p \leq 2\varepsilon^{1/p}\|s\|_\infty. \tag{2.2}$$

Since S is dense in $L^p(\mu)$, this completes the proof. □

Theorem 5 (Rudin 6.13)**.** *Suppose μ is a positive measure on $\mathfrak{M}$, $g \in L^1(\mu)$, and*

$$\lambda(E) = \int_E g \, d\mu \quad (E \in \mathfrak{M}). \tag{2.3}$$

Then

$$|\lambda|(E) = \int_E |g| \, d\mu \quad (E \in \mathfrak{M}). \tag{2.4}$$

Proof. By Theorem 2, there is a function h, of absolute value 1, such that $d\lambda = h \, d|\lambda|$. By hypothesis, $d\lambda = g \, d\mu$. Hence

$$h \, d|\lambda| = g \, d\mu.$$

This gives $d|\lambda| = \bar{h}g \, d\mu$. (Compare with Theorem 1.29.) Since $|\lambda| \geq 0$ and $\mu \geq 0$, it follows that $\bar{h}g \geq 0$ a.e. $[\mu]$, so that $\bar{h}g = |g|$ a.e. $[\mu]$. □ □

Theorem 6 (Rudin 6.16)**.** *Suppose $1 \leq p < \infty$, μ is a σ-finite positive measure on X, and Φ is a bounded linear functional on $L^p(\mu)$. Then there is a unique $g \in L^q(\mu)$, where q is the exponent conjugate to p, such that*

$$\Phi(f) = \int_X fg \, d\mu \quad (f \in L^p(\mu)). \tag{2.5}$$

Moreover, if Φ and g are related as in (1), we have

$$\|\Phi\| = \|g\|_q. \tag{2.6}$$

In other words, $L^q(\mu)$ is isometrically isomorphic to the dual space of $L^p(\mu)$, under the stated conditions.

Proof. Rudin 6.16: Duality of L^1 and L^∞ (not in Mathlib `https://leanprover.zulipchat.com/#narrow/channel/217875-Is-there-code-for-X.3F/topic/Lp.20duality/near/495207025`) □

Lemma 7. *Let X be a locally compact Hausdorff space, and let Φ be a bounded linear functional on $C_0(X)$. Suppose that μ, ν are regular complex Borel measure such that*

$$\Phi f = \int_X f \, d\mu = \int_X f \, d\nu.$$

Then $\mu = \nu$.

Proof. Suppose μ is a regular complex Borel measure on X and $\int f\,d\mu = 0$ for all $f \in C_0(X)$. By Theorem 2 there is a Borel function h, with $|h| = 1$, such that $d\mu = h\,d|\mu|$. For any sequence $\{f_n\}$ in $C_0(X)$ we then have

$$|\mu|(X) = \int_X (\bar{h} - f_n)h\,d|\mu| \leq \int_X |\bar{h} - f_n|\,d|\mu|, \tag{3}$$

and since $C_c(X)$ is dense in $L^1(|\mu|)$ (Theorem 4), $\{f_n\}$ can be so chosen that the last expression in (3) tends to 0 as $n \to \infty$. Thus $|\mu|(X) = 0$, and $\mu = 0$. It is easy to see that the difference of two regular complex Borel measures on X is regular. This shows that at most one μ corresponds to each Φ. ☐

Lemma 8. *Consider a given bounded linear functional Φ on $C_0(X)$. Assume $\|\Phi\| = 1$. (Update statement to be the general case.) We shall construct a positive linear functional Λ on $C_c(X)$, such that*

$$|\Phi(f)| \leq \Lambda(|f|) \leq \|f\| \quad (f \in C_c(X)), \tag{4}$$

where $\|f\|$ denotes the supremum norm.

Proof. Assume $\|\Phi\| = 1$, without loss of generality.

So all depends on finding a positive linear functional Λ that satisfies (4). If $f \in C_c^+(X)$ [the class of all nonnegative real members of $C_c(X)$], define

$$\Lambda f = \sup\{|\Phi(h)| : h \in C_c(X), |h| \leq f\}. \tag{9}$$

Then $\Lambda f \geq 0$, Λ satisfies (4), $0 \leq f_1 \leq f_2$ implies $\Lambda f_1 \leq \Lambda f_2$, and $\Lambda(cf) = c\Lambda f$ if c is a positive constant. We have to show that

$$\Lambda(f + g) = \Lambda f + \Lambda g \quad (f \text{ and } g \in C_c^+(X)), \tag{10}$$

and we then have to extend Λ to a linear functional on $C_c(X)$.

Fix f and $g \in C_c^+(X)$. If $\varepsilon > 0$, there exist h_1 and $h_2 \in C_c(X)$ such that $|h_1| \leq f$, $|h_2| \leq g$, and

$$\Lambda f \leq |\Phi(h_1)| + \varepsilon, \quad \Lambda g \leq |\Phi(h_2)| + \varepsilon. \tag{11}$$

There are complex numbers α_i, $|\alpha_i| = 1$, so that $\alpha_i \Phi(h_i) = |\Phi(h_i)|$, $i = 1, 2$. Then

$$\begin{aligned}
\Lambda f + \Lambda g &\leq |\Phi(h_1)| + |\Phi(h_2)| + 2\varepsilon && (2.7)\\
&= \Phi(\alpha_1 h_1 + \alpha_2 h_2) + 2\varepsilon && (2.8)\\
&\leq \Lambda(|h_1| + |h_2|) + 2\varepsilon && (2.9)\\
&\leq \Lambda(f + g) + 2\varepsilon, && (2.10)
\end{aligned}$$

so that the inequality $\geq$ holds in (10).

Next, choose $h \in C_c(X)$, subject only to the condition $|h| \leq f + g$, let $V = \{x : f(x) + g(x) > 0\}$, and define

$$h_1(x) = \frac{f(x)h(x)}{f(x) + g(x)}, \quad h_2(x) = \frac{g(x)h(x)}{f(x) + g(x)} \quad (x \in V), \tag{12}$$

$$h_1(x) = h_2(x) = 0 \quad (x \notin V). \tag{2.11}$$

It is clear that h_1 is continuous at every point of V. If $x_0 \notin V$, then $h(x_0) = 0$; since h is continuous and since $|h_1(x)| \leq |h(x)|$ for all $x \in X$, it follows that x_0 is a point of continuity of h_1. Thus $h_1 \in C_c(X)$, and the same holds for h_2.

Since $h_1 + h_2 = h$ and $|h_1| \leq f$, $|h_2| \leq g$, we have

$$|\Phi(h)| = |\Phi(h_1) + \Phi(h_2)| \leq |\Phi(h_1)| + |\Phi(h_2)| \leq \Lambda f + \Lambda g. \tag{2.12}$$

Hence $\Lambda(f+g) \leq \Lambda f + \Lambda g$, and we have proved (10).

If f is now a real function, $f \in C_c(X)$, then $2f^+ = |f| + f$, so that $f^+ \in C_c^+(X)$; likewise, $f^- \in C_c^+(X)$; and since $f = f^+ - f^-$, it is natural to define

$$\Lambda f = \Lambda f^+ - \Lambda f^- \quad (f \in C_c(X), f \text{ real}) \tag{13}$$

and

$$\Lambda(u + iv) = \Lambda u + i\Lambda v. \tag{14}$$

Simple algebraic manipulations, just like those which occur in the proof of Theorem 1.32, show now that our extended functional Λ is linear on $C_c(X)$. □

Theorem 9 (Rudin 6.19)**.** *If X is a locally compact Hausdorff space, then every bounded linear functional Φ on $C_0(X)$ is represented by a regular complex Borel measure μ, in the sense that*

$$\Phi f = \int_X f \, d\mu \tag{1}$$

for every $f \in C_0(X)$.

Proof. Once we have the Λ from Lemma 8, we associate with it a positive Borel measure λ, as in Theorem 2.14. The conclusion of Theorem 2.14 shows that λ is regular if $\lambda(X) < \infty$. Since

$$\lambda(X) = \sup\{\Lambda f : 0 \leq f \leq 1, f \in C_c(X)\} \tag{2.13}$$

and since $|\Lambda f| \leq 1$ if $\|f\| \leq 1$, we see that actually $\lambda(X) \leq 1$.

We also deduce from (4) that

$$|\Phi(f)| \leq \Lambda(|f|) = \int_X |f| \, d\lambda = \|f\|_1 \quad (f \in C_c(X)). \tag{5}$$

The last norm refers to the space $L^1(\lambda)$. Thus Φ is a linear functional on $C_c(X)$ of norm at most 1, with respect to the $L^1(\lambda)$-norm on $C_c(X)$. There is a norm-preserving extension of Φ to a linear functional on $L^1(\lambda)$, and therefore Theorem 6 (the case $p = 1$) gives a Borel function g, with $|g| \leq 1$, such that

$$\Phi(f) = \int_X fg \, d\lambda \quad (f \in C_c(X)). \tag{6}$$

Each side of (6) is a continuous functional on $C_0(X)$, and $C_c(X)$ is dense in $C_0(X)$. Hence (6) holds for all $f \in C_0(X)$, and we obtain the representation (1) with $d\mu = g \, d\lambda$. □

Lemma 10 (Rudin 6.19)**.** *Moreover, the norm of Φ is the total variation of μ:*

$$\|\Phi\| = |\mu|(X). \tag{2}$$

Proof. Since $\|\Phi\| = 1$, (6) shows that

$$\int_X |g| \, d\lambda \geq \sup\{|\Phi(f)| : f \in C_0(X), \|f\| \leq 1\} = 1. \tag{7}$$

We also know that $\lambda(X) \leq 1$ and $|g| \leq 1$. These facts are compatible only if $\lambda(X) = 1$ and $|g| = 1$ a.e. $[\lambda]$. Thus $d|\mu| = |g| \, d\lambda = d\lambda$, by Theorem 5, and

$$|\mu|(X) = \lambda(X) = 1 = \|\Phi\|, \tag{8}$$

which proves (2). □

Theorem 11. *Placeholder to combine the three results which make up The Reisz Theorem.*

Proof. □

Chapter 3

Orthogonal projections

- Destination: Mathlib.Analysis.InnerProductSpace.Projection
- Principal reference: Chapter 12 of [Walter Rudin, Functional Analysis.][Rud87].

Let H be a complex Hilbert space and K be a closed subspace of H. We denote $K^\perp$ the orthogonal complement of K in H. Any vector $x \in H$ can be written as $x = x_K + x_{K^\perp}$, where $x_K \in K, x_{K^\perp} \in K^\perp$. The map $p(K) : x \to x_K$ is called the orthogonal projection to K.

Lemma 12. *It holds that $p(K) = p(K)^2 = p(K)^*$.*

Proof. The first equality follows by the uniqueness of the orthogonal decomposition.

The second equality follows because $\langle y, p(K)x\rangle = \langle y, x_K\rangle = \langle y_K, x\rangle = \langle p(K)y, x\rangle$ by orthogonality. □

Lemma 13. *For $p \in \mathcal{B}(H)$ such that $p = p^2 = p^*$, there is a closed subspace K such that $p = p(K)$.*

Proof. By $p = p^2$, it is a projection. Let K be the image of p. Note that $x = (p + (1-p))x = px + (1-p)x$ for any $x \in H$ and $\langle px, (1-p)x\rangle = \langle x, (p-p)x\rangle = 0$. So this gives the orthogonal decomposition. □

Lemma 14 (Rudin 12.6, part 1)**.** *Let $\{x_n\}$ be a sequence of pairwise orthogonal vectors in H. Then the following are equivalent.*

- *$\sum_{n=1}^\infty x_n$ converges in the norm topology of H.*
- *$\sum_{n=1}^\infty \|x_n\|^2 < \infty$.*

Proof. Note that, by orthogonality, $\|\sum_{j=m}^n x_j\|^2 = \sum_{j=m}^n \|x_j\|^2$. Therefore, the second condition implies that the sequence $\sum_{j=1}^n x_j$ is Cauchy.

Conversely, as $\sum_{j=1}^n x_j$ converges in norm, the square of its norm $\sum_{j=1}^n \|x_j\|^2$ converges. □

Lemma 15 (Rudin 12.6, part 2)**.** *Let $\{x_n\}$ be a sequence of pairwise orthogonal vectors in H. Then the following are equivalent.*

- *$\sum_{n=1}^\infty x_n$ converges in the norm topology of H.*
- *$\sum_{n=1}^\infty \langle x, y\rangle$ converges for all $y \in H$.*

Proof. The first condition implies the second by Cauchy-Schwartz.

Assume that $\sum_{n=1}^{\infty}\langle x, y\rangle$ converges for all $y \in H$. Define $\Lambda_n y = \sum_{j=1}^{n}\langle y, x_j\rangle$. As this converges for each y, by Banach-Steinhaus, $\{\|\Lambda_n\|\}$ is bounded. As $\|\Lambda_n\| = \sqrt{\sum_{j=1}^{n}\|x_j\|}$, this gives the first condition. $\square$

Chapter 4

Resolutions of the identity

- Destination: ?
- Principal reference: Chapter 12 of [Walter Rudin, Functional Analysis.][Rud87].

Definition 16 (Rudin 12.17)**.** Let $\mathfrak{M}$ be a σ-algebra in a set Ω, and let H be a Hilbert space. For simplicity, we assume that Ω is a locally compact (Hausdorff) space. In this setting, a *resolution of the identity* (on $\mathfrak{M}$) is a mapping

$$E : \mathfrak{M} \to \mathfrak{M}(H)$$

with the following properties:

1. $E(\emptyset) = 0$, $E(\Omega) = I$.
2. Each $E(\omega)$ is a self-adjoint projection.
3. $E(\omega' \cap \omega'') = E(\omega')E(\omega'')$.
4. If $\omega' \cap \omega'' = \emptyset$, then $E(\omega' \cup \omega'') = E(\omega') + E(\omega'')$.
5. For every $x \in H$ and $y \in H$, the set function $E_{x,y}$ defined by:

$$E_{x,y}(\omega) = (E(\omega)x, y)$$

is a complex regular Borel measure on $\mathcal{M}$.

Lemma 17. *For any $x \in H$,*

$$E_{x,x}(\omega) = (E(\omega)x, x) = \|E(\omega)x\|^2.$$

Proof. □

Lemma 18. *For any $x \in H$, $E_{x,x}$ is a positive measure on $\mathfrak{M}$ whose total variation is:*

$$\|E_{x,x}\| = E_{x,x}(\Omega) = \|x\|^2.$$

Proof.

□

Lemma 19. *For two ω_1, ω_2, $E(\omega_1), E(\omega_2)$ commute.*

Proof. By (3), any two of the projections $E(\omega)$ commute with each other. □

Lemma 20. *If $\omega' \cap \omega'' = \emptyset$, then the ranges of $E(\omega')$ and $E(\omega'')$ are orthogonal to each other*

Proof. By (1), (3) and Theorem 12.14. □

Lemma 21. *If $\{\omega_j\}$ is a finite family of mutually disjoint Borel sets, then $E(\bigcup_j \omega_j) = \sum_j E(\omega_j)$.*

Proof. By (4) and induction. □

Remark: $\sum_{n=1}^{\infty} E(\omega_n)$ does not converge in the norm topology of $\mathcal{B}(H)$.

Lemma 22. *Let $x \in H$ and $\{\omega_j\}$ be a countable family of mutually disjoint Borel sets. Then $E(\bigcup_j \omega_j)x = \sum_j E(\omega_j)x$, where the right-hand side converges in the norm topology of H.*

Proof. Since $E(\omega_n)E(\omega_m) = 0$ when $n \neq m$, the vectors $E(\omega_n)x$ and $E(\omega_m)x$ are orthogonal to each other (Theorem 12.14). By (5),

$$\sum_{n=1}^{\infty} (E(\omega_n)x, y) = (E(\omega)x, y) \tag{4.1}$$

for every $y \in H$. It now follows from Theorem 14 that:

$$\sum_{n=1}^{\infty} E(\omega_n)x = E(\omega)x.$$

The series ((4.1)) converges in the norm topology of H. □

Proposition 23 (Rudin 12.18)**.** *If E is a resolution of the identity, and if $x \in H$, then*

$$\omega \mapsto E(\omega)x$$

is a countably additive H-valued measure on $\mathfrak{M}$.*

Proof. This is the summary of what is proved above. □

Moreover, sets of measure zero can be handled in the usual way:

Proposition 24 (Rudin 12.19)**.** *Suppose E is a resolution of the identity. If $\omega_n \in \mathfrak{M}$ and $E(\omega_n) = 0$ for $n = 1, 2, 3, ...$, and if*

$$\omega = \bigcup_{n=1}^{\infty} \omega_n,$$

then $E(\omega) = 0$.

Proof. Since $E(\omega_n) = 0$, $E_{x,x}(\omega_n) = 0$ for every $x \in H$. Since $E_{x,x}$ is countably additive, it follows that $E_{x,x}(\omega) = 0$. But

$$\|E(\omega)x\|^2 = E_{x,x}(\omega).$$

Hence, $E(\omega) = 0$. □

Chapter 5

The Spectral Theorem

Functional Analysis by Walter Rudin 1991, extract from Chapter 12

It should perhaps be stated explicitly that the spectrum $\sigma(T)$ of an operator $T \in \mathcal{B}(H)$ will always refer to the full algebra $\mathcal{B}(H)$. In other words, $\lambda \in \sigma(T)$ if and only if $T - \lambda I$ has no inverse in $\mathcal{B}(H)$. Sometimes we shall also be concerned with closed subalgebras A of $\mathcal{B}(H)$ which have the additional property that $I \in A$ and $T^* \in A$ whenever $T \in A$. (Such algebras are sometimes called $*$-algebras.)

Let A be such an algebra, and suppose that $T \in A$ and $T^{-1} \in \mathcal{B}(H)$. Since TT^* is self-adjoint, $\sigma(TT^*)$ is a compact subset of the real line (Theorem 12.15), hence does not separate $\mathbb{C}$, and therefore $\sigma_A(TT^*) = \sigma(TT^*)$, by the corollary to Theorem 10.18. Since TT^* is invertible in $\mathcal{B}(H)$, this equality shows that $(TT^*)^{-1} \in A$, and therefore $T^{-1} = T^(TT^*)^{-1}$ is also in A.

Thus T has the same spectrum relative to all closed *-algebras in $\mathcal{B}(H)$ that contain T.

Theorem 12.23 will be obtained as a special case of the following result, which deals with normal algebras of operators rather than with individual ones.

Theorem 25 (12.22)**.** *If A is a closed normal subalgebra of $\mathcal{B}(H)$ which contains the identity operator I and if Δ is the maximal ideal space of A, then the following assertions are true:*

1. *There exists a unique resolution E of the identity on the Borel subsets of Δ which satisfies*

$$T = \int_\Delta \widehat{T}\, dE \tag{5.1}$$

   for every $T \in A$, where $\widehat{T}$ is the Gelfand transform of T.

2. *The inverse of the Gelfand transform (i.e., the map that takes $\widehat{T}$ back to T) extends to an isometric *-isomorphism of the algebra $L^\infty(E)$ onto a closed subalgebra B of $\mathcal{B}(H)$, $B \supset A$, given by*

$$\Phi f = \int_\Delta f\, dE \quad (f \in L^\infty(E)). \tag{5.2}$$

   Explicitly, Φ is linear and multiplicative and satisfies

$$\Phi(\bar{f}) = (\Phi f)^*, \|\Phi f\| = \|f\|_\infty \quad (f \in L^\infty(E)).$$

3. *B is the closure [in the norm topology of $\mathcal{B}(H)$] of the set of all finite linear combinations of the projections $E(\omega)$.*

4. *If $\omega \subset \Delta$ is open and nonempty, then $E(\omega) \neq 0$.*

5. *An operator $S \in \mathcal{B}(H)$ commutes with every $T \in A$ if and only if S com mutes with every projection $E(\omega)$.*

Proof. Recall that (5.1) is an abbreviation for

$$(Tx, y) = \int_{\Delta} \widehat{T}\, dE_{x,y} \quad (x, y \in H, T \in A). \tag{5.3}$$

Since $\mathcal{B}(H)$ is a B^*-algebra (Section 12.9), our given algebra A is a commutative B^*-algebra. The Gelfand-Naimark theorem 11.18 asserts therefore that $T \to \widehat{T}$ is an isometric *-isomorphism of A onto $C(\Delta)$.

This leads to an easy proof of the uniqueness of E. Suppose E satisfies (5.3). Since $\widehat{T}$ ranges over all of $C(\Delta)$, the assumed regularity of the complex Borel measures $E_{x,y}$ shows that each $E_{x,y}$ is uniquely determined by (5.3); this follows from the uniqueness assertion that is part of the Riesz representation theorem ([23], Th. 6.19) 11. Since, by definition, $(E(\omega)x, y) = E_{x,y}(\omega)$, each projection $E(\omega))$ is also uniquely determined by (5.3).

This uniqueness proof motivates the following proof of the existence of E. If $x \in H$ and $y \in H$, Theorem 11.18 shows that $\widehat{T} \mapsto (Tx, y)$ is a bounded linear functional on $C(\Delta)$, of norm $\leq \|x\|\|y\|$, since $\|\widehat{T}\|_\infty = \|T\|$. The Riesz representation theorem supplies us therefore with unique regular complex Borel measures $\mu_{x,y}$ on Δ such that

$$(Tx, y) = \int_{\Delta} \widehat{T}\, d\mu_{x,y} \quad (x, y \in H, T \in A). \tag{5.4}$$

For fixed T, the left side of (5.4) is a bounded sesquilinear functional on H, hence so is the right side, and it remains so if the continuous function $\widehat{T}$ is replaced by an arbitrary bounded Borel function f. To each such f corresponds therefore an operator $\Phi f \in \mathcal{B}(H)$ (see Theorem 12.8) such that

$$((\Phi f)x, y) = \int_{\Delta} f\, d\mu_{x,y} \quad (x, y \in H). \tag{5.5}$$

Comparison of (5.4) and (5.5) shows that $\Phi\hat{T} = T$. Thus Φ is an extension of the inverse of the Gelfand transform.

It is clear that Φ is linear.

Part of the Gelfand-Naimark theorem states that T is self-adjoint if and only if $\hat{T}$ is real-valued. For such T,

$$\int_{\Delta} \widehat{T}\, d\mu_{x,y} = (Tx, y) = (x, Ty) = \overline{(Ty, x)} = \overline{\int_{\Delta} \hat{T} d\mu_{y,x}},$$

and this implies that $\mu_{y,x} = \overline{\mu_{x,y}}$. Hence,

$$((\Phi\bar{f})x, y) = \int_{\Delta} \bar{f}\, d\mu_{x,y} = \overline{\int_{\Delta} f\, d\mu_{y,x}} = \overline{((\Phi f)y, x)} = (x, (\Phi f)y)$$

for all $x, y \in H$, so that

$$\Phi\bar{f} = (\Phi f)^*. \tag{5.6}$$

Our next objective is the equality

$$\Phi(fg) = (\Phi f)(\Phi g) \tag{5.7}$$

for bounded Borel functions f, g on Δ. If $S \in A$ and $T \in A$, then $(ST)^\wedge = \widehat{S}\widehat{T}$; hence

$$\int_\Delta \hat{S}\hat{T}\, d\mu_{x,y} = (STx, y) = \int_\Delta \widehat{S}\, d\mu_{Tx,y}.$$

This holds for every $\widehat{S} \in C(\Delta)$; hence the two integrals are equal if $\widehat{S}$ is replaced by any bounded Borel function f. Thus

$$\int_\Delta f\widehat{T} d\mu_{x,y} = \int_\Delta f\, d\mu_{Tx,y} = ((\Phi f)Tx, y) = (Tx, z) = \int_\Delta \hat{T} d\mu_{x,z},$$

where we put $z = (\Phi f)^* y$. Again, the first and last integrals remain equal if $\widehat{T}$ is replaced by g. This gives

$$\begin{aligned}(\Phi(fg)x, y) &= \int_\Delta fg\, d\mu_{x,y} = \int_\Delta g\, d\mu_{x,z} \\ &= ((\Phi g)x, z) = ((\Phi g)x, (\Phi f)^* y) = (\Phi(f)\Phi(g)x, y),\end{aligned}$$

and (5.7) is proved.

We are finally ready to define E: If ω is a Borel subset of Δ, let χ_ω be its characteristic function, and put

$$E(\omega) = \Phi(\chi_\omega).$$

By (5.7), $E(\omega \cap \omega') = E(\omega)E(\omega')$. With $\omega' = \omega$, this shows that each $E(\omega)$ is a projection. Since Φf is self-adjoint when f is real, by (5.6), each $E(\omega)$ is self-adjoint. It is clear that $E(\emptyset) = \Phi(0) = 0$. That $E(\Delta) = I$ follows from (5.4) and (5.5). The finite additivity of E is a consequence of (5.5), and, for all $x, y \in H$,

$$E_{x,y}(\omega) = (E(\omega)x, y) = \int_\Delta \chi_\omega\, d\mu_{x,y} = \mu_{x,y}(\omega).$$

Thus (5.5) becomes (5.2). That $\|\Phi f\| = \|f\|_\infty$ follows now from Theorem 12.21.

This completes the proof of (1) and (2).

Part (3) is now clear because every $f \in L^\infty(E)$ is a uniform limit of simple functions (i.e., of functions with only finitely many values).

Suppose next that ω is open and $E(\omega) = 0$. If $T \in A$ and $\widehat{T}$ has its support in ω, (5.1) implies that $T = 0$; hence $\widehat{T} = 0$. Since $\widehat{A} = C(\Delta)$, Urysohn's lemma implies now that $\omega = \emptyset$. This proves (4).

To prove (5), choose $S \in \mathcal{B}(H)$, $x \in H$, $y \in H$, and put $z = S^* y$. For any $T \in A$ and any Borel set $\omega \subset \Delta$ we then have

$$(STx, y) = (Tx, z) = \int_\Delta \widehat{T}\, dE_{x,z}, \tag{5.8}$$

$$(TSx, y) = \int_\Delta \hat{T}\, dE_{Sx,y}, \tag{5.9}$$

$$(SE(\omega)x, y) = (E(\omega)x, z) = E_{x,z}(\omega),$$

$$(E(\omega)Sx, y) = E_{Sx,y}(\omega).$$

If $ST = TS$ for every $T \in A$, the measures in (5.8) and (5.9) are equal, so that $SE(\omega) = E(\omega)S$. The same argument establishes the converse. $\square$